Basic Functions

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$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

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Polynomials

Definition Polynomial is an expression of the type

$$P = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

where the coefficients $a_0, a_1, \ldots, a_n$ are real numbers and $a_n \neq 0$.

The polynomial P is of degree n.

A number x for which P(x)=0 is called a root of the polynomial P.

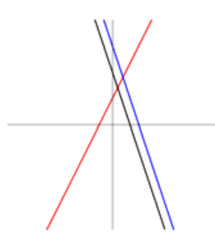
Theorem A polynomial of degree n has at most n real roots. Polynomials may have no real roots, but a polynomial of an odd degree has always at least one real root.

Graphs of Linear Polynomials

Graphs of linear polynomials $y = ax + b$ are straight lines. The coefficient "a" determines the angle at which the line intersects the x –axis.

Graphs of the linear polynomials:

1. $y = 2x+1$ (the red line)
2. $y = -3x+2$ (the black line)
3. $y = -3x + 3$ (the blue line)



Graphs of Higher Degree Polynomials

The behaviour of a polynomial $P = a_0 + a_1x + \ldots + a_nx^n$ for large positive or negative values x is determined by the highest degree term "a_nx^n".

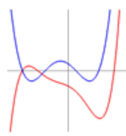
If $a_n > 0$ and n is odd, then as $x \to \infty$ also $P(x) \to \infty$.

Likewise: as $x \to -\infty$ also $P(x) \to -\infty$.

If $a_n > 0$ and n is even, then as $x \to \pm\infty$, $P(x) \to \infty$.

Problem

The picture on the right shows the graphs and all roots of a 4th degree polynomial and of a 5th degree polynomial. Which is which?



Solution

The blue curve must be the graph of the 4th degree polynomial because of its behavior as *x* grows or gets smaller.

Exponential Functions

Exponential functions are functions of the form

$$f(x) = a^x.$$

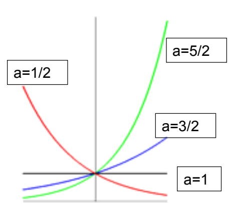
Assuming that $a > 0$, a^x is a well defined expression for all $x \in \mathbb{R}$.

The picture on the right shows the graphs of the functions:

1) $y = \left(\frac{1}{2}\right)^x$, the red curve
2) $y = 1^x$, the black line
3) $y = \left(\frac{3}{2}\right)^x$, the blue curve, and
4) $y = \left(\frac{5}{2}\right)^x$, the green curve.

The Number e

From the picture it appears obvious that, as the parameter *a* grows, also the slope of the tangent, at $x = 0$, of the graph of the function a^x grows.



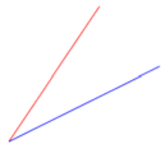
Definition

The mathematical constant e is defined as the unique number e for which the slope of the tangent of the graph of e^x at $x = 0$ is 1.

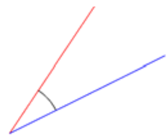
$e \approx 2.718281828$

Measuring of Angles (1)

Angles are formed by two half-lines starting from a common vertex. One of the half-lines is the starting side of the angle, the other one is the ending side. In this picture the starting side of the angle is blue, and the red line is the ending side.



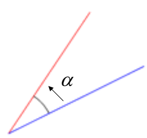
Angles are measured by drawing a circle of radius 1 and with center at the vertex of the angle. The size, in radians, of the angle in question is the length of the black arc of this circle as indicated in the picture.



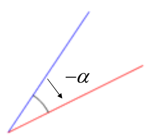
In the above we have assumed that the angle is oriented in such a way that when walking along the black arc from the starting side to the ending side, then the vertex is on our left.

Measuring of Angles (2)

The first picture on the right shows a positive angle.



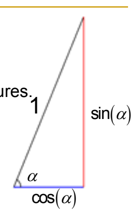
The angle becomes negative if the orientation gets reversed. This is illustrated in the second picture.



This definition implies that angles are always between -2π and 2π. By allowing angles to rotate more than once around the vertex, one generalizes the concept of angles to angles greater than 2π or smaller than -2π.

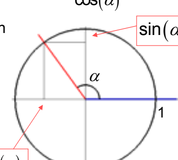
Trigonometric Functions (1)

Consider positive angles α, as indicated in the pictures.



Definition

The quantities $\sin(\alpha)$ and $\cos(\alpha)$ are defined by placing the angle α at the origin with starting side on the positive x-axis. The intersection point of the end side and the circle with radius 1 and with center at the origin is $(\cos(\alpha), \sin(\alpha))$.



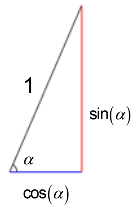
This definition applies for positive angles. We extend that to the negative angles by setting

$$\sin(-\alpha) = -\sin(\alpha) \text{ and}$$
$$\cos(-\alpha) = \cos(\alpha).$$

Trigonometric Functions (2)

$$\sin^2(\alpha)+\cos^2(\alpha)=1$$

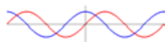
This basic identity follows directly from the definition.



Definition $\tan(\alpha)=\dfrac{\sin(\alpha)}{\cos(\alpha)}$ $\cot(\alpha)=\dfrac{\cos(\alpha)}{\sin(\alpha)}$

Graphs of:

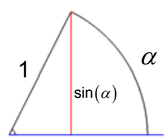
1. sin(x), the red curve, and
2. cos(x), the blue curve.



Trigonometric Functions (3)

The size of an angle is measured as the length α of the arc, indicated in the picture, on a circle of radius 1 with center at the vertex.

On the other hand, sin(α) is the length of the red line segment in the picture.



Lemma For positive angles α, $\sin(\alpha)\le\alpha$.

The above inequality is obvious by the above picture. For negative angles α the inequality is reversed.

Trigonometric Functions (4)

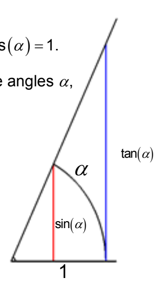
Trigonometric functions $\sin(\alpha)$ and $\cos(\alpha)$ are everywhere continuous, and $\lim_{\alpha\to 0}\sin(\alpha)=0$ and $\lim_{\alpha\to 0}\cos(\alpha)=1$.

In view of the picture on the right, we have, for positive angles α,

$$\sin(\alpha)\le\alpha\le\tan(\alpha).$$

Hence $1\le\dfrac{\alpha}{\sin(\alpha)}\le\dfrac{1}{\cos(\alpha)}.$

This implies: $\lim_{\alpha\to 0+}\dfrac{\sin(\alpha)}{\alpha}=1$



Lemma $\lim_{\alpha\to 0}\dfrac{\sin(\alpha)}{\alpha}=1$

Examples

Problem 1 Compute $\lim_{x\to 0}\frac{\sin(2x)}{x}$.

Solution Rewrite $\frac{\sin(2x)}{x}=2\left(\frac{\sin(2x)}{2x}\right)$.

By the previous Lemma, $\lim_{x\to 0}\frac{\sin(2x)}{2x}=1$.

Hence $\frac{\sin(2x)}{x}=2\left(\frac{\sin(2x)}{2x}\right)\xrightarrow[x\to 0]{}2$.

Examples

Problem 2 Compute $\lim_{x\to 0}\frac{\sin(\sin(x))}{x}$.

Solution Rewrite $\frac{\sin(\sin(x))}{x}=\frac{\sin(\sin(x))}{\sin(x)}\frac{\sin(x)}{x}$.

By the previous Lemma, $\lim_{x\to 0}\frac{\sin(\sin(x))}{\sin(x)}=1$.This follows by substituting $\alpha=\sin(x)$. As $x\to 0$, also $\alpha\to 0$.

Hence $\frac{\sin(\sin(x))}{x}=\frac{\sin(\sin(x))}{\sin(x)}\frac{\sin(x)}{x}\xrightarrow[x\to 0]{}1$.

Trigonometric Identities 1

Defining Identities

$$\csc(\alpha)=\frac{1}{\sin(\alpha)}\qquad \sec(\alpha)=\frac{1}{\cos(\alpha)}\qquad \cot(\alpha)=\frac{1}{\tan(\alpha)}$$

$$\tan(\alpha)=\frac{\sin(\alpha)}{\cos(\alpha)}\qquad \cot(\alpha)=\frac{\cos(\alpha)}{\sin(\alpha)}$$

Derived Identities

$$\sin(-\alpha)=-\sin(\alpha)\quad \cos(-\alpha)=\cos(\alpha)$$
$$\sin(\alpha+2\pi)=\sin(\alpha)\quad \cos(\alpha+2\pi)=\cos(\alpha)$$
$$\sin^2(\alpha)+\cos^2(\alpha)=1$$
$$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$$
$$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$$

Trigonometric Identities 2

Derived Identities (cont'd)

$$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$$
$$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$$
$$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \qquad \tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

$$\cos(2x)=\cos^2(x)-\sin^2(x) \qquad \sin(2x)=2\sin(x)\cos(x)$$
$$\cos(2x)=2\cos^2(x)-1 \qquad \cos(2x)=1-2\sin^2(x)$$
$$\cos^2(x)=\frac{1+\cos(2x)}{2} \qquad \sin^2(x)=\frac{1-\cos(2x)}{2}$$

Trigonometric Identities 3

Derived Identities (cont'd)

$$\sin(x)\cos(y)=\frac{1}{2}\left(\sin(x+y)+\sin(x-y)\right)$$
$$\cos(x)\cos(y)=\frac{1}{2}\left(\cos(x+y)+\cos(x-y)\right)$$
$$\sin(x)\sin(y)=\frac{1}{2}\left(\cos(x-y)-\cos(x+y)\right)$$